Delayed-feedback control of oscillations
in non-linear planar systems

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Abstract

The use of delayed feedback of position for the control of oscillatory behaviour is
analysed. It is shown that linear feedback can annihilate limit cycles and stabilize
the origin. Furthermore, it is possible to create an asymptotically stable limit cycle
with any prescribed amplitude using a simple nonlinear feedback. For almost all val-
ues of the delay, these feats can be achieved without using derivative information in
the feedback. The results indicate a possible mechanism by which biological systems
accomplish complex control tasks using delayed and partial state information.

1 Introduction

Feedback delays are an inevitable feature of many natural and man-made control mech-
anisms. While often looked at as an undesired characteristic, which can destabilize the
system or at best complicate the analysis, positive uses of delays have also been studied.
These go back to at least about half a century ago (Tallman and Smith 1958), followed
by more recent works in the 1980s (Suh and Bien 1979, Suh and Bien 1980, Shanmu-
gathasan and Johnston 1988, Kwon et al. 1990), where delays were used to enhance the
system performance in various ways. Most of the analytical studies have so far focused
on linear systems and stability. The present paper addresses the investigation of feedback
laws to control the amplitude and stability of oscillations in planar systems with general
nonlinearities.

The object of study of this paper is systems described by equations of the form

\[ \ddot{x} + \omega^2 x + \varepsilon g(x, \dot{x}, \varepsilon) = \varepsilon f(x(t - \tau)), \]

where \( x \in \mathbb{R} \), and \( \omega \) and \( \varepsilon \) are positive parameters. The left hand side describes the
dynamics of a planar system, or alternatively, that of a higher-dimensional system after
reducing the dynamics onto a two-dimensional centre manifold corresponding to a pair of
imaginary eigenvalues \( \pm j\omega \). The right hand side of (1) represents a feedback of position
which is delayed by \( \tau \geq 0 \). The feedback is scaled by the parameter \( \varepsilon \) for convenience
in following the effects of the nonlinearity \( g \). Systems of the form (1) with a positive
delay arise in diverse biological and industrial settings, for example in the production of
proteins (Hastings et al. 1977, an der Heiden 1979), orientation control in the fly (Reichardt
and Poggio 1976, Poggio and Reichardt 1976), neuromuscular regulation of movement and
posture (an der Heiden 1979, Beuter et al. 1993, Eurich and Milton 1996), acousto-optical
bistability (Vallée et al. 1987), metal cutting (Berger et al. 1993), vibration absorption
(Olgac and Holm-Hansen 1994), and control of an inverted pendulum (Atay 1999). The
presence of only the position information in the feedback is typical in many biological control mechanisms. Furthermore, the classical control-theoretic approach of using an observer to reconstruct the derivative is clearly not an option in natural systems. Hence, it is an interesting and challenging goal to discover the theoretical basis for control under partial and delayed information.

Control of oscillations is a practically important problem in many applications (Fradkov and Pogromsky 1998), which can generally be stated in terms of two different objectives:

(a) To obtain an asymptotically stable zero solution which attracts all initial conditions in a suitably large region (regulator problem).

(b) To obtain an asymptotically stable periodic solution with desired properties (such as oscillation at a given amplitude) and which attracts all initial conditions in a suitably large region (oscillator problem).

These requirements can be appropriately modified depending on the application; however, in general the desired action is more than simply stabilizing some particular trajectory. For instance, if the uncontrolled system has other periodic solutions, it may be necessary to annihilate them through the action of feedback in order to meet the objectives above. It is not sufficient to merely ensure that the unwanted periodic solutions are unstable, since their existence could severely limit the domain of attraction of the desired solution in the phase space, as well as degrade the general performance. Ideally, one might like to keep the phase-space dynamics as simple as possible, whereby every trajectory is attracted to the desired solution. While this may not always be a realistic goal, it makes it clear that the problem at hand is of a global nature.

The main result of this paper is that objective (a) above can be achieved by a linear delayed feedback of the variable $x$, and (b) can be achieved by a simple cubic feedback function. In addition, the amplitude of the periodic solution in (b) can be set arbitrarily. The conclusion holds for general nonlinearities $g$ and almost all positive values of the delay $\tau$, provided that $\varepsilon$ is not too large. Furthermore, these feats can be achieved using only the feedback of the position $x$, provided that $\sin \omega \tau \neq 0$. Thus, when the derivative $\dot{x}$ is not available for feedback, which is the case in many biological applications, the presence of a positive delay in (1) is crucial for the control of oscillations.

2 Periodic solutions and stability

We use $|| \cdot ||$ for the usual Euclidean norm, $j$ for the imaginary unit, and $D_i g$ for the partial derivative of the function $g$ with respect to its $i$th argument. Throughout the rest of the paper it will be assumed that $\omega = 1$ in (1), which can always be achieved by a rescaling of the time $t \mapsto \omega t$ and thus presents no loss of generality. Furthermore, $g$ will be assumed to satisfy the following hypothesis.

(H) The function $g : \mathbb{R}^3 \to \mathbb{R}$ is $C^2$, and $g(0,0,\varepsilon) = 0$ for all $\varepsilon$.

We shall investigate (1) using averaging theory. In the amplitude-phase variables $(r, \theta)$ defined by the transformation

\[
\begin{align*}
x(t) &= r(t) \cos(t + \theta(t)) \\
\dot{x}(t) &= -r(t) \sin(t + \theta(t)),
\end{align*}
\]  

(2)
(1) takes the form
\[
\dot{r} = \varepsilon \sin(t + \theta)(g - f) \\
\dot{\theta} = \varepsilon \frac{1}{r} \cos(t + \theta)(g - f)
\] (3)
where the arguments of \(f\) and \(g\) are expressed in terms of \(r\) and \(\theta\), i.e.
\[
g = g(r(t) \cos(t + \theta(t)), -r(t) \sin(t + \theta(t)), \varepsilon) \\
f = f(r(t - \tau) \cos(t - \tau + \theta(t - \tau)))
\] (4)

When \(\varepsilon = 0\), the solutions of (3) are constants, which correspond by (2) to the usual harmonic oscillations. Hence for small \(\varepsilon\), (3) can be viewed as a time-dependent perturbation of the harmonic oscillator in amplitude-phase variables, which can be analysed by the method of averaging. To this end, we briefly give an outline of averaging theory for functional differential equations. Denoting \(y = (r, \theta) \in \mathbb{R}^2\), the system (3)-(4) is a functional differential equation describing the relation between the instantaneous derivative \(\dot{y}(t)\) and the present and past values of \(y(t)\). Under appropriate continuity conditions on \(f\) and \(g\), a solution \(y(t)\) of (3) describes a trajectory in the state space \(\mathcal{C} = C([-\tau, 0], \mathbb{R}^2)\), the Banach space of continuous functions mapping the interval \([-\tau, 0]\) to \(\mathbb{R}^2\). The trajectory consists of points \(y_t\) which are defined by \(y_t(s) = y(t + s), s \in [-\tau, 0]\). Hence each point \(y_t \in \mathcal{C}\) on the trajectory corresponds to a “window” of the solution \(y(t)\) over an interval of length equal to the delay \(\tau\). In this notation, (3) has the form
\[
\dot{y}(t) = \varepsilon \bar{h}(t, y_t, \varepsilon)
\] (5)
where \(h\) is periodic in \(t\) with period \(T = 2\pi\). Letting
\[
\bar{h}(\varphi) = \frac{1}{T} \int_0^T h(t, \varphi, 0) \, dt,
\] (6)
the averaged equation corresponding to (5) is defined to be
\[
\dot{z}(t) = \varepsilon \bar{h}(z_t)
\] (7)
in which \(z_t\) is a constant element of \(\mathcal{C}\) (Hale 1966). Intuitively, \(y\) is slowly changing by (5), and so \(y_t(s) = y(t + s) + \mathcal{O}(\varepsilon)\) for \(s \in [-\tau, 0]\)—that is, \(y\) is almost a constant over an interval of length \(\tau\). In this way, averaging reduces the infinite-dimensional system (5) to an ordinary differential equation (7). We note that there are alternative ways of averaging which retain the delay term (Lehman and Weibel 1999), but the above method is particularly useful for the present analysis.

In stating the basic averaging result for (1) and equivalently for (3), we will refer to the following averaged quantities
\[
G(r) = \frac{1}{2\pi} \int_0^{2\pi} g(r \cos t, r \sin t, 0) \sin t \, dt
\] (8)
\[
F(r) = \frac{\sin \tau}{2\pi} \int_0^{2\pi} f(r \cos t) \cos t \, dt.
\] (9)
It is not hard to see that both \(G\) and \(F\) are odd functions of \(r\); i.e.
\[
G(-r) = -G(r) \quad \text{and} \quad F(-r) = -F(r) \quad \text{for all } r \in \mathbb{R}.
\] (10)
For instance,

\[ G(-r) = \frac{1}{2\pi} \int_0^{2\pi} g(-r \cos t, -r \sin t, 0) \sin t \, dt \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} g(r \cos(t + \pi), r \sin(t + \pi), 0) \cdot (-\sin(t + \pi)) \, dt \]

\[ = -\frac{1}{2\pi} \int_0^{3\pi} g(r \cos u, r \sin u, 0) \sin u \, du \]

\[ = -\frac{1}{2\pi} \int_0^{2\pi} g(r \cos u, r \sin u, 0) \sin u \, du \]

\[ = -G(r) \]

and similarly for \( F \).

The starting point in the analysis of (1) is provided by the following theorem, which shows that the dynamics of (1) is determined by \( G \) and \( F \) for sufficiently small \( \varepsilon \).

**Theorem 1** Assume the hypothesis (H), let \( f: \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) function with \( f(0) = 0 \), and let \( G \) and \( F \) be defined by (8)-(9). Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) the following hold.

(i) The zero solution of (1) is asymptotically stable if \( F'(0) + G'(0) > 0 \), and unstable if \( F'(0) + G'(0) < 0 \).

(ii) Let \( R > 0 \) be such that \( F(R) + G(R) = 0 \) and \( F'(R) + G'(R) \neq 0 \). Then (1) has a periodic solution given by

\[ x(t) = R \cos t + O(\varepsilon), \quad (11) \]

which is orbitally asymptotically stable if \( F'(R) + G'(R) > 0 \), and unstable if \( F'(R) + G'(R) < 0 \).

The idea of the proof is to reduce the dynamics of (1) to the dynamics of the scalar equation

\[ \dot{r} = -\varepsilon(G(r) + F(r)). \quad (12) \]

via averaging. The averaging theorem then relates positive and hyperbolic equilibrium points of (12) to nontrivial hyperbolic periodic solutions of (1) with the same stability type.

**Proof.** (i) A characteristic root \( \lambda \) of the linearization of (1) about the zero solution satisfies \( \Delta(\lambda, \varepsilon) = 0 \), where

\[ \Delta(\lambda, \varepsilon) = \lambda^2 + 1 + \varepsilon(D_1 g(0,0,\varepsilon) + \lambda D_2 g(0,0,\varepsilon)) - \varepsilon f'(0)e^{-\lambda \tau}. \quad (13) \]

When \( \varepsilon = 0 \), there are two roots on the imaginary axis: \( \lambda = \pm j \). An application of the implicit function theorem shows that the roots depend smoothly on \( \varepsilon \) in a neighborhood of \( \varepsilon = 0 \), and implicit differentiation of (13) gives

\[ \text{Re}[\lambda'(\varepsilon)]_{\varepsilon=0} = -\frac{1}{2}(f'(0) \sin \tau + D_2 g(0,0,0)) = -(F'(0) + G'(0)). \]

Thus, the roots \( \lambda \) move into the left (respectively, right) half-plane if \( F'(0) + G'(0) \) is positive (resp., negative), and remain there for all sufficiently small \( \varepsilon \).
(ii) We average the equation for \( r \) given in (3) in the sense of (6):

\[
\dot{r} = \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \sin(t + \theta) g(r \cos(t + \theta), -r \sin(t + \theta), 0) \, dt - \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \sin(t + \theta) f(r \cos(t - \tau + \theta)) \, dt.
\]

(14)

Here, in accordance with (7), \( r(t - \tau) \) and \( \theta(t - \tau) \) are replaced by \( r(t) \) and \( \theta(t) \), respectively, while \( r \) and \( \theta \) are treated as constants over one period, which is justified by the fact that these are slowly varying quantities if \( \varepsilon \) is small (Hale 1966). With the change of variable \( u = -(t + \theta) \) and using the fact that the integrand is \( 2\pi \)-periodic in \( u \), the first integral in (14) becomes

\[
\varepsilon \frac{1}{2\pi} \int_{-\theta}^{-\theta - 2\pi} (\sin u) g(r \cos u, r \sin u, 0) \, du = -\varepsilon G(r).
\]

Similarly, with \( u = t - \tau + \theta \), the second integral in (14) can be written as

\[
-\varepsilon \frac{1}{2\pi} \int_{-\theta}^{2\pi + \theta - \tau} \sin(u + \tau) f(r \cos u) \, du
= -\varepsilon \frac{\sin \tau}{2\pi} \int_0^{2\pi} f(r \cos u) \cos u \, du - \varepsilon \frac{\cos \tau}{2\pi} \int_0^{2\pi} f(r \cos u) \sin u \, du
\]

(15)

where we have used the fact that the second integral in (15) is zero. Hence the averaged equation for \( r \) is given by (12). If \( G(R) + F(R) = 0 \) for some \( R > 0 \), then \( R \) is an equilibrium of (12), and the corresponding eigenvalue is \( -\varepsilon (G'(R) + F'(R)) \). Thus, \( R \) is asymptotically stable (resp., unstable) if \( G'(R) + F'(R) \) is positive (resp., negative). Statement (ii) then follows by the averaging theorem for delay differential equations (Hale 1966) and the transformation (2).

Using Theorem 1, we shall address the objectives (a) and (b) of Section 1 in the next two sections.

3 Linear feedback

It is often of interest how much could be achieved by a linear feedback law. Hence consider the special case when \( f \) has the form

\[ f(x) = k_1 x, \]

(16)

for some \( k_1 \in \mathbb{R} \). Then by (9),

\[ F(r) = \frac{1}{2} r k_1 \sin \tau, \]

(17)

and the following result is an easy consequence of Theorem 1.

**Proposition 1** Assume the hypothesis (H), and let \( f \) be given by (16). Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) the following hold.

(i) The zero solution of (1) is asymptotically stable if \( k_1 \sin \tau > -D_2 g(0, 0, 0) \), and unstable if \( k_1 \sin \tau < -D_2 g(0, 0, 0) \).

(ii) Define the continuous function

\[
\bar{G}(r) = \begin{cases} 
G(r)/r & \text{if } r \neq 0, \\
G'(0) & \text{if } r = 0,
\end{cases}
\]

(18)
Let $R > 0$ be such that $\bar{G}(R) \neq 0$, and let $k_1 \sin \tau = -2\bar{G}(R)$. Then (1) has a periodic solution of the form (11), which is orbitally asymptotically stable (resp., unstable) if $\bar{G}(R)$ is positive (resp., negative).

**Proof.** Statement (i) is immediate by part (i) of Theorem 1. If $k_1 \sin \tau = -2\bar{G}(R)$, then $F(R) + G(R) = 0$, and

$$F'(R) + G'(R) = -\frac{G(R)}{R} + G'(R) = R\bar{G}'(R).$$

Statement (ii) then follows by part (ii) of Theorem 1. □

Thus, linear feedback can always stabilize the zero solution, and modify the amplitudes of limit cycles to some degree, provided $\sin \tau \neq 0$. However, the stability of periodic solutions are seen to depend only on $g$; and furthermore, there may be a multitude of such solutions, which may be undesirable in some applications. So, linear feedback is helpful in solving the oscillator problem only to the extent allowed by the nonlinearity $g$ (for an example see Atay 1998). On the other hand, it is possible to annihilate all periodic solutions inside an arbitrarily large ball through linear feedback, and make every trajectory inside converge to zero, i.e. obtain semiglobal stability. This can be seen by the following result.

**Proposition 2** Let $c > 0$, let $M_c > 0$ be such that

$$|G(r)| \leq M_c|r| \quad \text{if } |r| \leq c,$$

(such an $M_c$ exists by the hypothesis (H)), and suppose $k_1 \sin \tau > 2M_c$. If $\varepsilon > 0$ and $r(t)$ is a solution of the averaged equation (12) with $|r(0)| \leq c$, then $|r(t)| \leq c$ for all $t > 0$, and $r(t) \to 0$ as $t \to \infty$.

**Proof.** By the definition (8) of $G$ and the hypothesis (H) it is clear that $G$ is a differentiable function and $G(0) = 0$. Hence a positive $M_c$ satisfying (19) exists. Now suppose $k_1 \sin \tau > 2M_c$. Then by (17) $F''(0) > M_c > 0$, and by (19) $|G'(0)| \leq M_c$, so that $G''(0) + F'(0) > 0$, and the zero solution of (1) is asymptotically stable by Theorem 1. Furthermore, for $r \in [0, c]$,

$$F(r) + G(r) = \frac{1}{2}rk_1 \sin \tau + G(r) > M_cr - |G(r)| \geq 0$$

by (17) and (19). It follows that $F(r) + G(r)$ is negative for $r \in [-c, 0)$, since it is an odd function of $r$. Hence, the vector field of the averaged equation (12) points towards the origin at each point in $[-c, c] - \{0\}$, from which the statement of the proposition follows. □

Hence, objective (a) of Section 1 can be achieved by linear feedback. We illustrate through an example.

**Example.** Consider the equation

$$\ddot{x} - 0.8 \sin(\dot{x}) + x = 0. \quad (20)$$

We take $\varepsilon = 0.8$ and $g(x, \dot{x}, \varepsilon) = -\sin(\dot{x})$. The corresponding function $G(r)$ has infinitely many zeros, as seen in figure 1. Thus, (20) has an infinite number of (stable and unstable) periodic solutions, and the zero solution is unstable. Two of these solutions with widely different amplitudes, obtained through the numerical
simulation of (20), are shown in figure 2. Differentiating under the integral in (8) yields the estimate

\[ |G'(r)| = \frac{1}{2\pi} \int_0^{2\pi} |\cos(r \sin t) \sin^2 t| \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2}, \]

implying that

\[ |G(r)| \leq \frac{1}{2} |r|. \]

Application of Proposition 2 with any \( c > 0 \) and \( M_c = \frac{1}{2} \) shows that it is possible to annihilate the periodic solutions and stabilize the origin in the controlled system

\[ \ddot{x} - 0.8 \sin(\dot{x}) + x = 0.8k_1 x(t - \tau). \]  

(21)

provided \( k_1 \sin \tau > 1 \). The numerical solutions of (21) depicted in figure 2 confirm this observation, where it is seen that the same initial conditions which lead to the periodic solutions in the uncontrolled system decay to zero under the action of delayed feedback. Note that the domain of attraction of the origin is large.

We remark that even when \( c = \infty \) in the estimate (19), Proposition 2 cannot be used to conclude that the zero solution of (1) attracts all trajectories in the phase space. However, it is true that the radius of attraction tends to \( \infty \) as \( \varepsilon \) approaches zero (Guckenheimer and Holmes 1983).

### 4 Nonlinear feedback

The results of Section 3 indicate that linear feedback is sufficient for the regulator problem, but in general not for the oscillator problem. Hence, one is led to consider nonlinear
Figure 2: Two of the many periodic solutions of the uncontrolled system ($k_1 = 0$) which are annihilated by linear delayed feedback with gain $k_1 = 1.1$. The other parameter values are $\varepsilon = 0.8$ and $\tau = \pi/2$. 

feedback schemes. We show that objective (b) can be realized by the addition of a single nonlinear term to the feedback function.

**Proposition 3** Assume the hypothesis (H), let \( R > 0 \), and suppose \( \sin \tau \neq 0 \). Then there exists \( \varepsilon_0 > 0 \) and a feedback function of the form

\[
f(x) = k_1 x + k_3 x^3
\]

such that for each \( \varepsilon \in (0, \varepsilon_0) \), (1) has an orbitally asymptotically stable periodic solution whose amplitude is \( R + O(\varepsilon) \).

The basic idea of the proof is to show that the coefficients \( k_i \) in (22) can be chosen so that the averaged equation (12) has a unique equilibrium point at \( R \) in some bounded (but possibly large) interval, which attracts all initial conditions there.

**Proof.** When \( f \) is given by (22), the corresponding function \( F(r) \) is also a cubic polynomial

\[
F(r) = q_1 r + q_3 r^3,
\]

with

\[
q_1 = \frac{1}{2} k_1 \sin \tau \quad \text{and} \quad q_3 = \frac{3}{8} k_3 \sin \tau.
\]

Consequently,

\[
F(r) + G(r) = r(q_1 + q_3 r^2 + \tilde{G}(r)),
\]

where the function \( \tilde{G} \) is given by (18). Now fix \( c > R > 0 \). It is easy to see that \( \tilde{G} \) is continuously differentiable on \( R \); so, the following nonnegative numbers exist:

\[
\mu_1 = \max\{|\tilde{G}(r)| : r \in [0, c]\},
\]

\[
\mu_2 = \max\{|\tilde{G}'(r)| : r \in [0, c]\}.
\]

Now let \( R > 0 \), choose \( q_1 \) such that

\[
q_1 < -\frac{5}{3} \mu_1 - R \mu_2,
\]

and define \( q_3 \) by

\[
q_3 = -q_1 - \frac{\tilde{G}(R)}{R^2} > 0.
\]

Note from (24) that positive roots of \( F + G \) and of the function

\[
H(r) := q_1 + q_3 r^2 + \tilde{G}(r)
\]

coincide. Using (28), one has

\[
H(r) = q_1 \left(1 - \frac{r^2}{R^2}\right) + \tilde{G}(r) - \frac{r^2}{R^2} \tilde{G}(R),
\]

so, \( H(R) = 0 \). We claim that \( H \) has no other positive roots in \( (0, c) \). Indeed, for \( r \in (0, R/2] \),

\[
H(r) \leq q_1 \left(1 - \frac{r^2}{R^2}\right) + \mu_1 \left(1 + \frac{r^2}{R^2}\right)
\]

\[
\leq \frac{3}{4} q_1 + \frac{5}{4} \mu_1 < -\frac{3}{4} R \mu_2 \leq 0,
\]
where the last inequality follows by (27). Hence, \( H(r) \) has no roots on the interval \((0, R/2]\).

On the other hand, for \( r \in [R/2, c] \),
\[
    H'(r) = \frac{-2(q_1 + G(R))}{R^2} r + G'(r) \geq \frac{-2(q_1 + \mu_1) R}{2} - \mu_2 \\
    \geq \frac{1}{R} \left( -q_1 - \frac{5}{3} \mu_1 - R \mu_2 \right) > 0,
\]
(30)

Thus \( H \) is strictly increasing for \( r \in [R/2, c] \), so that any root in this interval is necessarily unique. Hence, \( R \) is the unique root of \( H \) (and consequently of \( F + G \)) on the interval \((0, c]\).

Furthermore,
\[
    F'(R) + G'(R) = H(R) + RH'(R) = 0 + RH'(R) > 0
\]
by (30), and
\[
    F'(0) + G'(0) = H(0) = q_1 + G(0) \leq q_1 + \mu_1 < 0,
\]
by (24) and (27). So, by Theorem 1, the origin is unstable and there is an asymptotically stable periodic solution whose amplitude is \( R + O(\varepsilon) \). Using (23), the coefficients of (22) are found as \( k_1 = 2q_1(\sin \tau)^{-1} \) and \( k_3 = 8q_3(3 \sin \tau)^{-1} \).

The proof suggests a simple procedure to design a nonlinear feedback to obtain a periodic solution with amplitude \( R \). Using (23), (27), and (28), the coefficient \( k_3 \) in the cubic feedback function (22) can be determined from the knowledge of \( k_1 \) through the formula
\[
    k_3 = -\frac{8}{3R^2} \left( \frac{1}{2} k_1 + \frac{G(R)}{R \sin \tau} \right).
\]
(31)

Furthermore, by Proposition 1 one needs to ensure
\[
    k_1 \sin \tau < -D_2 g(0, 0, 0) = -2G'(0)
\]
(32)
to prevent the solutions from decaying to zero. In this way, the design of the feedback law is reduced to the tuning of a single gain parameter, \( k_1 \). An example will serve to illustrate the use of nonlinear feedback.

**Example.** Consider the example of Section 3 once more, this time with \( \varepsilon = 1.5 \) and a cubic feedback:
\[
    \ddot{x} - 1.5 \sin(\dot{x}) + x = 1.5 \left( k_1 x(t - \tau) + k_3 (x(t - \tau))^3 \right).
\]
(33)

Suppose it is desired to have stable limit cycle oscillations with amplitude \( R = 2 \). With \( g(x, \dot{x}, \varepsilon) = -\sin(\dot{x}) \) as before, one has \( D_2 g(0, 0, 0) = -1 \), and a numerical calculation gives \( G(2) = -0.577 \). The parameter values \( \tau = \pi/2 \) and \( k_1 = 0.6 \) satisfy (32), and determining \( k_3 = -7.67 \times 10^{-3} \) from (31) one obtains the limit cycle shown in figure 3. Other trajectories starting from a variety of initial conditions converge to the limit cycle, shown by dotted lines in the same figure. Hence, the domain of attraction of the limit cycle is reasonably large. The extent of the domain depends in general on the values of \( k_1, \varepsilon, \) and \( \tau \), and increases with decreasing \( \varepsilon \) when other parameters are fixed.

In closing this section, we note that if \( f \) is an even function, then (9) gives \( F(-r) = F(r) \), so that, in view of (10), one has \( F(r) \equiv 0 \). Hence, \( F \) only depends on the odd part \( f_o(x) = \frac{1}{2}(f(x) - f(-x)) \) of \( f \). Since Theorem 1 implies that the dynamics of (1) is determined by \( G \) and \( F \) only, there is no loss of generality in assuming that \( f \) is an odd function. In this sense, (22) represents the simplest nonlinear feedback function (at least in the ring of polynomials). Together with the results of the previous section, it is seen that simple delayed feedback schemes can be quite powerful in oscillation control.
Figure 3: The limit cycle with amplitude 2, shown with the solid line, obtained for the parameter values $\varepsilon = 1.5$, $\tau = \pi/2$, $k_1 = 0.6$, and $k_3 = -7.67 \times 10^{-3}$. The dotted lines are trajectories starting from arbitrary initial conditions and converging to the limit cycle.

5 Note on perturbation magnitude

The conclusions of any perturbation analysis are rigorously justified for sufficiently small values of the perturbation parameter. However, the results may also remain valid for larger perturbations. In fact, in the examples (21) and (33) the value of $\varepsilon$ was 0.8 and 1.5, respectively, which make the nonlinearity and the feedback about the same order as the remaining terms. This shows that the upper limit $\varepsilon_0$ appearing in the statement of Propositions 1 and 3 need not be small. The actual value of $\varepsilon_0$ depends on the other parameters and the nature of the nonlinearities of the system. In particular, there is an important relation between $\varepsilon_0$ and the necessary condition $\sin \tau \neq 0$ that appear in the statement of Propositions 1 and 3. Intuitively, if delayed position feedback is not useful when $\sin \tau = 0$, then it is expected to have limited power when $\sin \tau$ is close to zero. This is indeed the case, the reason being that the allowable perturbation magnitude $\varepsilon_0$ diminishes as $\sin \tau$ approaches zero.

To justify this observation, we consider the problem from a bifurcation point of view. When $\varepsilon = 0$ (1) has a pair of purely imaginary eigenvalues; thus, one expects to see a Hopf bifurcation as $\varepsilon$ is increased past zero. The averaging theory provides a convenient tool for analysing the bifurcating periodic solutions. As $\varepsilon$ continues to increase away from zero further bifurcations may occur. However, these are not reflected in the averaged equations since the qualitative behaviour of (12) remains the same for all $\varepsilon > 0$. In such a situation the conclusions obtained by averaging lose their validity. The multitude of possible local and global bifurcations prevent making a general statement for arbitrary nonlinearities;
nevertheless, the problem can be demonstrated for the example system

\[ \ddot{x} - \varepsilon \sin(\dot{x}) + x = \varepsilon k_1 x(t - \tau). \] (34)

The characteristic equation (13) in this case is

\[ \Delta(\lambda, \varepsilon) = \lambda^2 + 1 - \varepsilon \lambda - \varepsilon k_1 e^{-\lambda \tau}. \] (35)

Consider a stationary bifurcation, i.e. a zero eigenvalue crossing the imaginary axis, as a possible mechanism for altering the dynamical behaviour that was obtained for small \( \varepsilon \). Substituting \( \lambda = 0 \) into (35) gives \( \varepsilon k_1 = 1 \). Hence, a stationary bifurcation is not possible as long as

\[ 0 < \varepsilon < \frac{1}{k_1}. \] (36)

On the other hand, if the aim is to stabilize the zero solution, then by Proposition 1 one should have \( k_1 \sin \tau > -D_2 g(0, 0, 0) = 1 \). Assuming \( \sin \tau > 0 \) for simplicity, this condition can be written as \( k_1 > 1/\sin \tau \). In view of (36) the implication is

\[ \varepsilon < \sin \tau. \] (37)

In other words, if \( \varepsilon \) is larger than \( \sin \tau \) a stationary bifurcation occurs which may invalidate the stability results obtained for small \( \varepsilon \). Figure 4 gives some numerically calculated values of maximum \( \varepsilon \) for which (34) is stable for some \( k_1 \). The dependence on \( \tau \) is seen to follow the curve \( \sin \tau \) closely, in direct support of the analytical estimation (37). In particular, the allowable perturbation magnitude reaches a peak at about \( \tau = \pi/2 \) and approaches zero as \( \tau \to 0 \). Both (37) and figure 4 make clear the relevance of the condition \( \sin \tau \neq 0 \) to the perturbation bound \( \varepsilon_0 \).

Several remarks are in order. Firstly, (37) pertains to the stability of the origin and not the other periodic solutions. For instance, in the example of Section 4 a stable periodic solution was obtained when \( \varepsilon \) was as large as 1.5. Secondly, in arriving at (37) we have considered only stationary bifurcations. The transcendental equation (35) can also undergo a Hopf bifurcation—in fact a sequence of such bifurcations. So (37) should be viewed not as a general formula but only an estimate, which is valid for a certain range of parameters. Finally, recall the change of time scale \( t \to \omega t \) which allowed taking \( \omega = 1 \) in (1). In the original time scale of (1), the necessary condition for delayed position feedback to work becomes \( \sin \omega \tau \neq 0 \).

6 Conclusion

We have shown that delayed feedback of position can be effectively used in the control of oscillatory behaviour. In view of previous studies, the local effects of linear delayed feedback, such as stabilizing the zero solution, are perhaps not surprising. The results presented in this paper rigorously show that linear feedback is also capable of more global deeds like annihilating all the periodic solutions and other limit sets in a large domain, thus ensuring that trajectories inside converge to zero. Moreover, by adding a single cubic term to the feedback function, it is possible to create stable limit cycle oscillations with any prescribed amplitude. The necessary condition \( \sin \omega \tau \neq 0 \) implies that these feats cannot be accomplished by undelayed feedback of position, exhibiting another positive use of delays in control. It also provides a possible explanation of how biological systems achieve control tasks in the absence of full state information and without the use of asymptotic observers.
Figure 4: The upper bound $\varepsilon_0$ of the allowable perturbation magnitude $\varepsilon$ for stability of the system (34), calculated for various values of the delay $\tau$. The numerical estimations closely follow the curve $\varepsilon_0 = \sin \tau$ over the range $\tau \in [0, \pi/2]$.

References


